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TRUNCATED SEQUENTIAL HYPOTHESIS TESTS

Julian J. Bussgang and Michael B. Marcus

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

In the operation of phased-array radars it is possible, in principle, to save power without sacrificing performance by not deciding in advance how long an observation should be made, e.g., through sequential testing. In practice, some limit must be put on the freedom of the designer to choose observation times--hence the test is truncated. This Memorandum describes and evaluates a class of such tests.

The work was undertaken as basic research in technology applicable to the design of electronically scanned radars of potential use in ballistic missile defenses. It is part of a continuing study for ARPA on low-altitude defense against ballistic missiles.

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SUMMARY

Through a careful examination of the equations by which Wald determines the values of the boundaries for tests of sequential hypotheses, we are able to obtain interesting relationships for the conditional probability distributions of the stage at which the test terminates. These enable us to study sequential tests in which the boundaries are functions of the sample number. We are particularly concerned with tests with convergent boundaries, and investigate a set of boundaries which approach those of a truncated Wald test. Approximate expressions for the expected sample number and probabilities of error are obtained for the tests considered. The obtained approximations apply best for gently tapering slopes. Extensions of our method can be applied to evaluate various *ad hoc* schemes for truncating tests and to the theory of tests with a varying parameter.

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CONTENTS

I. INTRODUCTION

The sequential test of statistical hypotheses as analyzed by Wald(1) entails two parallel boundaries, the crossing of each of which is associated with the acceptance of one of the two alternate hypotheses. Although a proof of Stein(2) provides assurance that these tests terminate with probability 1, in practice it is frequently desirable to truncate the sequential test at a pre-determined stage. The truncation can occur in many ways; in particular, it may be abrupt, i.e., no change in the test procedure is introduced until the truncation stage itself, or gradual, i.e., the test procedure is modified at every stage so that the boundaries monotonically converge.

A difficulty arises in the treatment of truncated sequential tests in that the boundaries are themselves a function of the sample size, which is a random variable. Wald(1) derived certain bounds on the probabilities of error of an abruptly truncated test, but they do not appear to be very tight.

The main purpose of this Memorandum is to determine some approximate relations between the truncated and untruncated tests. A class of gradually truncated tests is considered. Results are obtained for the Average Sample Number and for the probability of accepting an alternate hypothesis. The method employed is based on a detailed examination of the relationships used by Wald, and the hitherto unnoticed implied relationships for the conditional probability distributions of the stage at which the test terminates. The same method can be extended to obtain the Operating Characteristic and to generate better approximations.

II. FUNDAMENTAL RELATIONS

Following Wald[(1)] we consider sequential tests on the sequences $x_1, x_2, \ldots$ of identically and independently distributed random variables, where we know that the probability measure on the sample space is generated by one of two probability density functions, p_0 or p_1. It is the purpose of the sequential alternate hypothesis test to determine which of the density functions governs the observations. We shall use the standard measure on the infinite sample space (the extension of the product measure of finite subspaces)[(3)] and assume that the density functions are such that the likelihood ratios are continuous functions of $x_1, \ldots, x_n$ for all n.

The sequential test is performed as follows: at each stage in the selection of the sample the likelihood ratio is formed. This process is continued as long as

$$e^{f_0(m)} < \prod_{j=1}^{m} \frac{p_1(x_j)}{p_0(x_j)} < e^{f_1(m)}, \quad m = 1,2,\ldots,n-1 \tag{1}$$

and ceases at some stage n as soon as one of the inequalities is violated. For the sake of clarity in presentation, the terminal stage is denoted by n as distinguished from an arbitrary stage m; n is a random variable.

Let H_0 and H_1 represent the null and alternate hypotheses and let α and β be the errors of the first and second kind. We associate a violation of the lower inequality with the acceptance of H_0 and a violation of the upper inequality with the acceptance of H_1.

The functions $f_0(m)$ and $f_1(m)$ in (1) are assumed to be either constant, or monotonically non-decreasing and non-increasing, respectively. In either case the test terminates with probability 1 as long as $f_0(m)$ and $f_1(m)$ are bounded, which we assume to be the case. The object of this Memorandum is to study truncated tests; truncation occurs when, for some value of m, say N, $f_0(N) = f_1(N)$, since at this point one of the inequalities in (1) must be violated. (The assumption that the likelihood ratio is continuous precludes the possibility that both inequalities be violated simultaneously except for a set of paths of probability zero.) Notice that if $f_i(N) \neq 0$, i=0,1, we could accept a hypothesis when the likelihood ratio is less than unity, i.e., when the hypothesis is _a posteriori_ less likely. It appears reasonable to always accept the _a posteriori_ more likely hypothesis. Therefore we require that $f_0(N) = f_1(N) = 0$ if the two hypotheses are _a priori_ equally likely and the costs of incorrect decisions are equal. If the two hypotheses are not equally likely or if the test has different costs associated with the acceptance of each hypothesis, the value of $f_i(N)$ (i=0,1) at truncation would not necessarily be zero but would depend upon the ratio of _a priori_ probabilities and costs.

Let $\mu_0^*(n)$ and $\mu_1^*(n)$ be the probability measures, corresponding to H_0 and H_1, of the set of paths $x_1, x_2, \ldots, x_n$ which cause the lower inequality in (1) to be violated for the first time at the n^{th} stage. (The notation is similar to Wald's: the single and double asterisks indicate the condition that the process exceeds the lower and upper bounds, respectively. The subscript i=0,1 indicates which hypothesis is true.) Similarly $\mu_0^{**}(n)$ and $\mu_1^{**}(n)$ are these quantities with respect to the upper inequality. We define the following conditional probability density functions

$$p_i^{*}(n) = \frac{\mu_i^{*}(n)}{\sum_{n=1}^{\infty} \mu_i^{*}(n)} \;,\; p_i^{**}(n) = \frac{\mu_i^{**}(n)}{\sum_{n=1}^{\infty} \mu_i^{**}(n)} \quad (i=0,1) \tag{2}$$

where, for example, $p_1^{**}(n)$ is the probability that the test terminates at the n^{th} stage conditioned by the facts that H_1 is the true hypothesis and that the upper inequality is the one which is violated. Based on these conditional probability density functions, corresponding conditional expectations can be calculated. They are denoted by $E_0^{*}(n)$, $E_1^{*}(n)$, $E_0^{**}(n)$ and $E_1^{**}(n)$.

In the following we restrict our attention to the upper inequality, the violation of which results in the acceptance of H_1; similar results hold for the lower inequality resulting in the acceptance of H_0. Let $x_1,\ldots,x_n$ be a sample path that results in the violation of the upper inequality at the n^{th} stage. Then

$$p_1(x_1)\ p_1(x_2)\cdots p_1(x_n) \geqslant e^{f_1(n)}\ p_0(x_1)\ p_0(x_2)\cdots p_0(x_n) \tag{3}$$

Let S be the set of n-tuples for which the above inequality holds, conditioned by the fact that expression (1) was satisfied at the n-1 previous stages. Then

$$\int_S \prod_{i=1}^{n} p_1(x_i)\,d\mu(x_1,\ldots,x_n) \geqslant e^{f_1(n)} \int_S \prod_{i=1}^{n} p_0(x_i)\,d\mu(x_1,\ldots,x_n) \tag{4}$$

where μ is n-dimensional Lebesque measure. The existence of these integrals is easy to verify; Eq. (4) is equivalent to the following statement, which better suits our purposes,

$$\mu_1^{**}(n) \geqslant e^{f_1(n)} \mu_0^{**}(n) \tag{5}$$

Furthermore, we shall assume that the inequality in (5) is in fact an equality. The implications of this assumption will be discussed at the end of Section III. Wald makes the same assumption, which is often described as "neglecting the excess over the boundaries," when he lets the constant upper threshold A (i.e., $e^{f_1(n)} = A > 1$) be equal to $(1-\beta)/\alpha$. Since, in the Wald test $\sum_{n=1}^{\infty} \mu_1^{**}(n) = 1-\beta$ and $\sum_{n=1}^{\infty} \mu_0^{**}(n) = \alpha$, the analog of Eq. (4) is $1-\beta > A\,\alpha$. Setting $A = (1-\beta)/\alpha$ implies that

$$\mu_1^{**}(n) = A\,\mu_0^{**}(n) \tag{6}$$

for all n. Since all the quantities are positive this follows for the following reason: if, for any n, the left-hand side in Eq. (6) were greater than the right-hand side, the sums over all n would give $A < (1-\beta)/\alpha$, which contradicts the assumption that $A = (1-\beta)/\alpha$. The equation $\sum_{n=1}^{\infty} \mu_1^{**}(n) = 1-\beta$ should be obvious; this is simply the probability, given that H_1 is true, that the test gives the correct answer. Similarly, $\sum_{n=1}^{\infty} \mu_0^{**}(n) = \alpha$ is the probability of error when

H_0 is the true hypothesis.

With respect to the Wald test, the conditional probabilities given in Eq. (2) are

$$p_1^{**}(n) = \frac{\mu_1^{**}(n)}{1-\beta} \;, \quad p_0^{**}(n) = \frac{\mu_0^{**}(n)}{\alpha} \tag{7}$$

Thus we have the following lemma and its corollaries:

LEMMA 1:

Consider a Wald test (neglecting the excess over the boundaries) which leads to the acceptance of the alternate hypothesis (H_1). For such a test, the conditional probability density function that the test ends at the n^{th} stage, given that the null hypothesis (H_0) is true, is equal to the conditional probability density function that the test ends at the n^{th} stage, given that the alternate hypothesis (H_1) is true; i.e., $p_0^{**}(n) = p_1^{**}(n)$ for all n. (Similarly, in the case when H_0 is accepted, $p_0^{*}(n) = p_1^{*}(n)$.) (See footnote.)

Corollary 1.1

In a Wald test the conditional moments of the sample size, when they exist, are all equal, i.e.,

$$E_0^{**}(n^r) = E_1^{**}(n^r),\ E_0^{*}(n^r) = E_1^{*}(n^r)$$

Note: This symmetry relationship on the conditional distribution of n was first observed by Bussgang and Middleton (Ref. 4) under certain specific conditions on the probability density function of the logarithm of the probability ratio. The normal distribution was observed to satisfy these conditions.

Corollary 1.2

Referring to the general test defined in Eq. (1) under the assumption that Eq. (5) is an equality, the following equalities are established

$$E_0^{**}\left(e^{f_1(n)}\right) = \frac{1-\beta}{\alpha} \tag{8}$$

$$E_1^{**}\left(e^{-f_1(n)}\right) = \frac{\alpha}{1-\beta} \tag{9}$$

$$E_0^{**}\left(e^{f_1(n)}\right)E_1^{**}\left(e^{-f_1(n)}\right) = 1 \tag{10}$$

(Similar results are obtained when H_0 is accepted.)

Both the lemma and its corollaries hold if the observations x_1, x_2,... are not independent. The independence of the samples, assumed in Section II, is actually first necessary in the next Section in calculating the Average Sample Number.

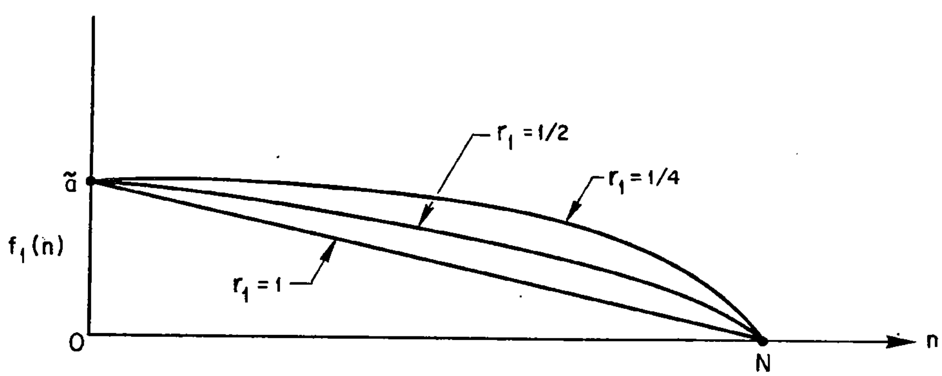
III. TESTS WITH GENTLY SLOPING BOUNDARIES

Consider the modified sequential alternate hypothesis test defined in expression (1) for which

$$f_0(n) = -\tilde{b}\,(1 - \frac{n}{N})^{r_0}$$

$$f_1(n) = \tilde{a}\,(1 - \frac{n}{N})^{r_1}$$

where $0 < r_0,\ r_1 \leq 1$ and $\tilde{a}$ and $\tilde{b}$ are positive. The graph of $f_1(n)$ is shown in the accompanying figure. In what follows the tilda sign (~) distinguishes the quantities characterizing the modified test from the corresponding quantities in the Wald test.



A class of upper boundaries of the truncated test

The graph of $f_0(n)$ is similar.

If we rewrite expression (1) for the logarithm of the likelihood ratio we obtain

$$-\widetilde{b}\ (1 - \frac{m}{N})^{r_0} \leq \sum_{j=1}^{m} \log \frac{p_1(x_j)}{p_0(x_j)} \leq \widetilde{a}\ (1 - \frac{m}{N})^{r_1} \tag{11}$$

$$m = 1,\ 2, \ldots,\ n-1$$

This is the equation which first interested the authors and provided the motivation for the relationships shown in Section II. In particular, if we consider the case where $p_i(x) = N(i,\sigma)$ (normal distribution with mean i and variance σ) then Eq. (11) defines a random walk (sums of independent random Gaussian-distributed variables) with convergent absorbing boundaries.

Let $\widetilde{E}_i(n)$ $(i=0,1)$ be the average test length for the test defined in Eq. (11), conditioned by the fact that H_i is the true hypothesis, and let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be the errors of the first and second kind for the test. We shall obtain approximate values for all of these quantities. Note that as $N \to \infty$, Eq. (11) defines the standard Wald test where $\widetilde{a} = \log A$ and $\widetilde{b} = -\log B$. We require that $\widetilde{a}/N$ and $-\widetilde{b}/N$ be small; this is what we mean by gently sloping boundaries, since in the $r_1 = 1$ case, $-\widetilde{a}/N$ is the slope of $f_1(n)$; in general $-r_1\widetilde{a}/N$ is the derivative of $f_1(n)$ at $n=0$. Similar results apply to $f_0(n)$ with $\widetilde{a}$ replaced by $\widetilde{b}$.

Since $\widetilde{a}/N$ and $-\widetilde{b}/N$ are small, the tests that we are examining are very close to the Wald test, whose boundary begins with the same value. They might be considered as Wald tests with slightly modified boundaries. This is significant since a meaningful interpretation of our results is obtained by comparing the class of tests discussed here to the Wald test with $\tilde{a} = a = \log A$, $\tilde{b} = b = -\log B$.

In the following we assume that $\tilde{\alpha}$ and $\tilde{\beta}$ are very small so that $1-\tilde{\alpha} \approx 1$ and $1-\tilde{\beta} \approx 1$. This assumption is not necessary but it greatly simplifies the resulting expressions. Following this assumption the equation

$$\tilde{E}_1(n) = \tilde{\beta}\, \tilde{E}_1^{\,*}(n) + (1-\tilde{\beta})\, \tilde{E}_1^{\,**}(n) \tag{12}$$

is replaced by

$$\tilde{E}_1(n) = \tilde{E}_1^{\,**}(n) \tag{13}$$

Now, setting $u = \frac{n}{N}$ and $z_j = \log \frac{p_1(x_j)}{p_0(x_j)}$, we use a well-known result of sequential analysis,[1,3] together with the often-mentioned neglect of excess over the boundaries, to obtain two equalities:

$$E_1(z_1 + \ldots\ldots + z_n) = E_1(n)\; E_1(z) \tag{14}$$

and

$$E_1(z_1 + \ldots\ldots + z_n) = \tilde{\beta}\, \tilde{E}_1^{\,*}\left[-\tilde{b}\,(1-u)^{r_0}\right] + (1-\tilde{\beta})\, \tilde{E}_1^{\,**}\left[\tilde{a}\,(1-u)^{r_1}\right] \tag{14a}$$

When $\tilde{\beta} << 1$, the first term on the right-hand side of (14a) can be ignored and by Eq. (13) we obtain

$$\tilde{E}_1^{\,**}(n)\; E_1(z) \approx \tilde{E}_1^{\,**}\left[\tilde{a}\,(1-u)^{r_1}\right]$$
$$\approx \tilde{E}_1^{\,**}\left\{\tilde{a}\left[1 - r_1 u + \frac{r_1(r_1-1)}{2}\, u^2 - \ldots\right]\right\}$$

Thus

$$\tilde{E}_1^{**}(n) \approx \tilde{E}_1(n) \geqslant \frac{\tilde{a}}{E_1(z) + \frac{r_1 \tilde{a}}{N}} \tag{15}$$

where we neglect all the conditional moments of u higher than the first. The reason for this will be discussed below.

In order to obtain $\tilde{\alpha}$ we use Eq. (9) with $f_1(n) = \tilde{a}\,(1-u)^{r_1}$, so that

$$\tilde{E}_1^{**}\left[e^{-\tilde{a}\,(1-u)^{r_1}}\right] \approx \frac{\tilde{\alpha}}{1-\tilde{\beta}} \tag{16}$$

Taking in account our approximation $1-\tilde{\beta} \approx 1$, neglecting the conditional moments of u higher than the first in the Taylor series expansion about u=0, and substituting Eq. (15) for $\tilde{E}_1^{**}(n)$, we get

$$\tilde{\alpha} \approx e^{-\tilde{a}}\left[1 + \frac{r_1 \tilde{a}^2}{N\,E_1(z) + r_1 \tilde{a}}\right] \tag{17}$$

Equations (15) and (17) apply when the null hypothesis is true, by replacing $\tilde{a}$ by $-\tilde{b}$, $\tilde{\alpha}$ by $\tilde{\beta}$ and $\tilde{E}_1(n)$ by $\tilde{E}_0(n)$.

Now consider the Wald test with upper boundary e^a and lower boundary e^{-b} and let α and β be the probabilities of error of the first and second kind. If α and β are very small, $E_1(n) \approx a/E_1(z)$ and $e^{-a} \approx \alpha$. Suppose $\tilde{a} = a$, i.e., the boundaries of the Wald test and the modified test begin (n=0) at the same points. We get from Eq. (15)

$$\tilde{E}_1(n) \geqslant \frac{E_1(n)}{1 + \frac{r_1}{N} E_1(n)} \tag{18}$$

and in place of Eq. (17) we have

$$\tilde{\alpha} \approx \alpha \left[1 + \frac{r_1 a \; E_1(n)}{N + r_1 \; E_1(n)} \right] \tag{19}$$

From Eq. (18) we see that $E_1(n)/(1+r_1) < \tilde{E}_1(n) \leqslant E_1(n)$; this is to be expected since, because of the truncation, the tests which we consider must have a shorter expected test length. However, because of the optimality of the Wald test the probability $\tilde{\alpha}$ must be greater than α, as it is (see Eq. (19)). In fact, if $\tilde{\alpha}$ were set equal to α we would expect the corresponding test with converging boundaries to have a greater expected length than the Wald test. This in turn implies also that the truncated test must begin at $\tilde{a} > a$.

In review, α is the probability of error of the first kind for a Wald test with upper boundary e^{a}. The probability $\tilde{\alpha}$ is the probability of error of the first kind for a test with upper boundary $\exp\left[\tilde{a}\,(1 - \frac{n}{N})^{r_1}\right]$, where we no longer assume that $a = \tilde{a}$ but assume $\alpha = \tilde{\alpha}$. It is immediately obvious that under those conditions $\tilde{a} > a$, since by (17)

$$\alpha = \tilde{\alpha} \approx e^{-\tilde{a}} \left[1 + \frac{r_1 \tilde{a}^2}{N \; E_1(z) + r_1 \tilde{a}} \right] \tag{20}$$

since $\alpha = e^{-a}$ and the term in the brackets is greater than 1.

Two additional questions of interest need clarification. First, in calculating the conditional moments of $u = \frac{n}{N}$ we disregard all moments except the first. Actually when we first obtained our results we included the first two moments and argued that the third moment could be neglected. This was because $u > 1$ so that $E_1^{**}(u) > E_1^{**}(u^2) > E_1^{**}(u^3)$, and further it is reasonable to believe that these inequalities are non-trivial. Also in the expression of the u^3 term in Eqs. (15) and (17) the coefficient is dominated by $1/(3!)$. Thus we satisfied ourselves that only the first two moments had to be included. To evaluate the equations including the first two moments we obtained an expression for $E_i(n^2)$ (i=0,1) from the two additional equations.

$$E_i\left[(z_1+\ldots\ldots+z_n)^2\right] = E_i^{**}(n^2)\left[E_i(z)\right]^2 + E_i^{**}(n)\left\{E_i(z^2) - \left[E_i(z)\right]^2\right\} \approx E_i^{**}\left[f_1^2(n)\right] \tag{21}$$

The expressions that we then obtained for Eqs. (15), (17), (18) and (19) were cumbersome and added no easy insight into the behavior of the tests. Thus we decided to display our results with only the first moment included. Greater accuracy can be obtained by including the second moments if this is desired.

By keeping only the first moment of u in Eq. (14) and in the results which follow, we are essentially approximating the boundaries of the converging test by straight lines. The implication is that the behavior of the test with linear boundaries corresponds to the behavior of the test with the actually specified boundaries. Retention of higher moments of u results in using higher-order polynomial approximations. It appears that there is no restriction on the order of polynomial approximation that can be tried (if the distribution of z, the logarithm of the likelihood ratio, is known) although the calculations grow increasingly more cumbersome.

The second question deals with the assumption that Eq. (5) is an equality. This is equivalent to assuming that for all the sample paths that result in the violation of expression (1) at the upper limit at the n^{th} stage, the equality is achieved. In other words, considering $e^{f_1(n)}$ as an upper boundary, we neglect the excess over the boundary. The validity of this assumption depends upon the properties of the stochastic processes generated by the density functions p_0 and p_1. If almost all the paths of the processes are continuous and if the upper and lower boundaries are continuous, then we can devise continuous tests for which this assumption is satisfied (see also Ref. 5). However, for specific density functions p_0 and p_1 not of this class one must decide the extent to which the results obtained in this Memorandum apply.

We note further that for the boundaries given at the beginning of this section, the assumption of equality in (5) becomes more strained as the boundaries converge, especially if the procedure cannot be approximated by a continuous procedure. If we let the parameters r_0 and r_1 approach zero, the test under consideration is the Wald test with a final decision at the N^{th} stage (determined by whether the likelihood ratio is greater or less than 1). However, at this stage there is clearly excess over the boundaries. Even by passing to continuous tests, our results do not apply when parallel boundaries are abruptly truncated because the boundaries are discontinuous at the last stage. That is why Eqs. (15), (17), (18) and (19) are meaningless for $r_0 = r_1 = 0$. These equations give the results for the Wald test, but that is because our assumption of neglecting the excess over the boundary results in our neglecting the stage at which truncation takes place.

A comment is in order on the closeness of the approximate equalities (15) and (17). In the case of $r_1 = 1$, we can compare our results with the exact results calculated by T. W. Anderson for a Wiener process.[6] For example, for $\tilde{a} = 3.981$, $N = 600.25$ and $E_1(z) = 0.02$, our result (Eq. (15)) yields $\tilde{E}_1(n) = 149$ rather than 139.2; for $\tilde{a} = 7.104$, $N = 870.26$ and $E_1(z) = 0.02$, the approximation (15) gives $\tilde{E}_1(n) = 252$ in place of 249.4. Our approximation for $\tilde{E}_1(n)$ appears therefore quite satisfactory in these cases. The approximate expression (17) gives, in the same two cases, $\tilde{\alpha} = 0.023$ and 0.0064 in place of 0.05 and 0.01. Since the right-hand side of (17) is a truncated expansion of an exponential series with a positive exponent, it is not surprising that (17) understates the value. The approximations improve as r_1 decreases. When $r_1 > 1$, the curvature of the boundaries is reversed, the initial slope is no longer gentle and expressions based on just the first two terms of a polynomial expansion will not, in general, suffice.

IV. CONCLUSIONS

In most applications of the sequential procedure it is desirable to set a definite upper limit for the number of observations. This can be achieved by changing the rules of procedure in such a way that by a certain stage N the acceptance and rejection regions meet, eliminating the "defer decision" region of the original sequential test of Wald. The original Wald procedure involves two parallel boundaries which, in the application of this procedure to detection problems, can be implemented as two fixed thresholds against which the detector output is compared. No decision is made as long as the detector output remains between the two thresholds.

Parallel boundaries are a consequence of assuming that the observations are independent, that the cost of taking each observation is the same, and that the time and hypothesized probability distributions remain unchanged throughout the test. If the cost of each observation were to increase as the test progressed, the test boundaries would monotonically converge. The class of modified test procedures analyzed in this Memorandum possesses this property. A practical interpretation of such boundaries is that the urgency to terminate the test becomes greater as the truncation point is approached. Another interpretation is that the absolute value of the logarithm of the likelihood ratio is expected to decrease with the progress of the test. This would occur if the two hypotheses converge.

The approximations made in this Memorandum imply that the truncation point is sufficiently beyond the Average Sample Number so that most tests terminate before the truncation point is reached. Thus events at the truncation point itself can, in fact, be ignored.

When the boundaries have no slope ($r=0$), the truncation stage N disappears from the equations and the test behaves as the original Wald test.

The general method of analysis presented in this Memorandum is to approximate non-constant boundaries by polynomials in the stage number. The approximation actually used here to illustrate the method is to retain only the first two terms of the binomial expansion of $(1-u)^r$. A further study is required to determine the best polynomial approximation of a given degree.

The modification of the boundaries changes not only the Average Sample Number but also the probabilities of error of the first and second kind. Specifically, the probabilities of error increase relative to a test whose boundaries begin at the same points and remain parallel. The probabilities of error of the modified test can be made to equal those of the unmodified test only if the boundaries of the former begin outside those of the latter.

It should be remarked that, in our opinion, the optimality of the sequential test is intimately related to the equality of conditional probabilities of terminating the test proved in Section II of this Memorandum. We have work in progress which attempts to develop a simple proof of the optimality of sequential tests based on this property of the Wald test.

The results presented in this Memorandum apply to cases when either one or both boundaries are gently sloped towards the truncation point. We have encountered this situation in the study of sequential estimation. Additional applications of these results should arise in problems of random walk since sequential tests can be regarded as a random walk with absorbing barriers.

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